Lecture 8

Dimension

We say a vector space V is said to be n-dimensional, written $\dim(V) = n$ if V has a basis with n ~~elemenst~~ elements

The vector space $\{0\}$ is defined to have dimension zero

$$\dim\left(\text{span}\begin{bmatrix}0\\0\\0\end{bmatrix}\right) = 0$$

standard basis

$$\dim(\mathbb{R}^2) = 2 \qquad \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$$

$$\dim(\mathbb{R}^3) = 3 \qquad \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$$

Lemma: suppose $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ spans V, then a set of $\underline{\underline{n+1}}$ or more vectors in V are linearly dependent

What is dim (rowsp(A))

$\Rightarrow$ The number of vectors in a basis for the rowsp(A)

We can take the matrix A and reduce to echelon form. The the number of non-zero rows is the number of elements in the basis and the row vectors form a basis.

Rank

rank(A) is the rank of matrix A and is equal to the max # of linearly independent rows of A, equivalently, the dim(rowsp(A))

Theorem

$$\dim(\text{rowsp}(A)) = \dim(\text{colsp}(A)) = \text{rank}(A)$$

2^{nd} basis example

Find a basis that is composed of the original vectors

Step 1: Form a matrix M whose columns are the given vectors

$$M = [\vec{u}_1^T, \vec{u}_2^T, \vec{u}_3^T, \vec{u}_4^T] = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 2 & 3 & 8 & 4 \\ 1 & 3 & 7 & 6 \\ 3 & 5 & 13 & 29 \end{bmatrix}$$

Step 2: Row reduce to echelon form

$$M \sim \begin{array}{c} \\ \left[\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \vec{u}_1^T \ \vec{u}_2^T \ \vec{u}_3^T \ \vec{u}_4^T \end{array}$$

no pivot

row equivalent

delete

Step 3: For each column without a pivot, delete (cast out) the vector $\vec{u}_k$ from the list

Step 4: Output the remaining vectors
basis : $\{\vec{u}_1, \vec{u}_2, \vec{u}_4\}$

$$\dim(\text{rowsp}(M)) = \dim(\text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_4\})$$

$$= \dim(\text{colsp}(M)) = \text{rank}(M) = 3$$

Some Examples

Consider

$$v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ -2 \\ -4 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -2 \\ 6 \\ 14 \end{bmatrix}$$

Are the vectors linearly dependent?

Method 1 :

- Construct a matrix composed of the vectors
- reduce to echelon form
- Count the # of non-zero rows

$\Rightarrow$ 2 non-zero rows $\quad A = [V_1, V_2, V_3]$

$\Rightarrow$ $\dim(\text{colsp}(A)) = 2$

$\Rightarrow$ A basis span $\{V_1, V_2 V_3\}$
has 2 elements $(n=2)$

$\Rightarrow$ 3 vectors $(n+1)$ is linearly dependent

Method 2:

Does there exist $k_1, k_2, k_3 \in \mathbb{R}$
s.t.

$$V_1 k_1 + V_2 k_2 + V_3 k_3 = 0 \; ?$$

If so, set is linearly dependent

$$\underbrace{[V_1, V_2, V_3]}_{A} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented matrix

$$\left[\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 0 & -2 & 6 & 0 \\ 1 & -4 & 14 & 0 \end{array}\right]$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{rank}(A) = 2$$

Check for consistency. If consistent then a solution exists, then the set is linearly dependent. For k_i not all zero

Linearly independent example

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

only solution is $k_2 = k_1 = 0$

the "trivial" solution

$\Rightarrow$ linearly independent set

~~What is the dim~~

Linear Mappings - Matrix Mappings

Generally two-types of mappings

non-square matrices : mapping from n-dimensional space to m-dimensional space

$$\begin{array}{c} \\ m \end{array} \underset{n}{\overset{A}{\begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}}} \underset{1}{\overset{\vec{x}}{\begin{bmatrix} \ \\ \ \\ \ \end{bmatrix}}} n = \overset{\vec{b}}{\begin{bmatrix} \ \\ \ \\ \ \end{bmatrix}} m$$

$$(m \times n)(n \times 1) = (m \times 1)$$

square matrices : refer to mapping as transformations
(e.g. reflection, rotation, dilation, translation)

In general let A be any $m \times n$ matrix, then A determines a mapping

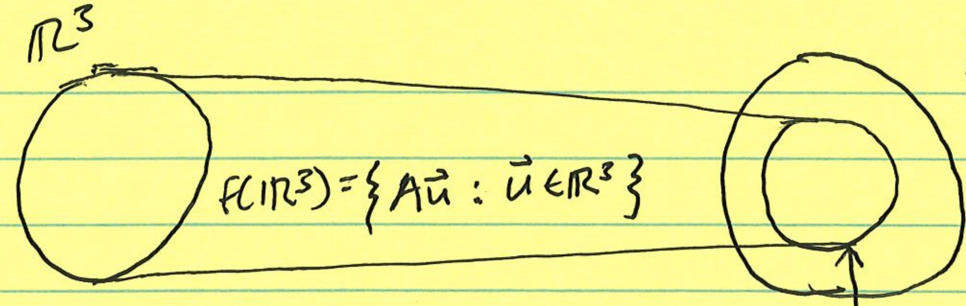
$$F_A : \mathbb{R}^n \to \mathbb{R}^m$$

where the vectors in $\mathbb{R}^m$ and $\mathbb{R}^n$ are written as columns

$$A = \begin{bmatrix} 5 & 1 & 0 \\ 4 & -1 & 1 \\ 3 & 0 & 8 \\ 0 & 1 & 2 \end{bmatrix} \qquad F_A : \mathbb{R}^3 \to \mathbb{R}^4$$

$$F_A(\vec{u}) = A\vec{u} \in \mathbb{R}^4$$

$\dim(\mathbb{R}^3) = 3$



"domain" of A

$f(\mathbb{R}^3)$ is called the image or range of f

$\dim(\text{colsp}(A)) = \dim(\text{Im}(A)) \leq 3$

image or range

$$f(\mathbb{R}^3) = \text{span}\left\{ \begin{bmatrix} 5 \\ 4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 8 \\ 2 \end{bmatrix} \right\}$$

* Note that I need 4 vectors a minimum to span $\mathbb{R}^4$

$\dim(\text{Im}(A))$

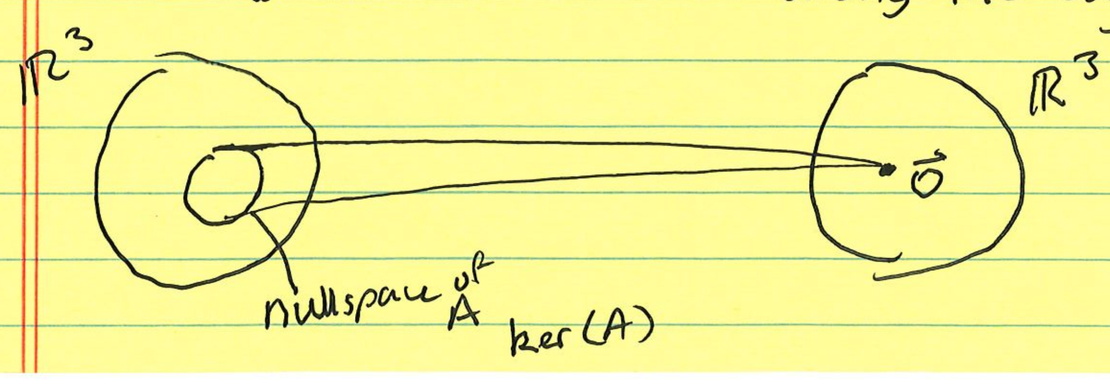
$\dim(f(\mathbb{R}^3)) = \dim(\text{Im}(A)) = 3$

Consider again vectors from H.W. P. 3

$$A = \begin{bmatrix} -1 & 0 & -2 \\ 0 & -2 & 6 \\ 1 & -4 & 14 \end{bmatrix} \Rightarrow \dim(\text{colsp}(A)) = 2$$

$\dim(\mathbb{R}^3) = 3$
"domain"

* We lost a dimension along the way



A different subspace of the "domain" called the kernel of A is the set of all vectors $\vec{v} \in \mathbb{R}^n$ s.t. $A\vec{v} = \vec{0}$. These vectors are in the nullspace of A

$$\boxed{\text{nullsp}(A) = \ker(A)}$$

<u>Rank and nullity of linear mapping</u>

$$\text{rank}(A) = \dim(\text{Im} A)$$
$$\text{nullity}(A) = \dim(\ker A)$$

<u>Theorem</u>: Let $\mathbb{R}^n$ be of finite dimension and let $F_A : \mathbb{R}^n \to \mathbb{R}^m$, the

$$\dim(\mathbb{R}^n) = \dim(\ker A) + \dim(\text{Im} A)$$
$$= \text{nullity}(A) + \text{rank}(A)$$

Ex 1 $A = \begin{bmatrix} 5 & 1 & 0 \\ 4 & -1 & 1 \\ 3 & 0 & 8 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\text{rank}(A) = 3$

$\dim(\mathbb{R}^3) = 3 \Rightarrow \dim(\ker A) = 0$

only the zero vector maps to zero!!!

Ex 2

$$A = \begin{bmatrix} -1 & 0 & -2 \\ 0 & -2 & 6 \\ 1 & -4 & 14 \end{bmatrix}$$

$\text{rank}(A) = 2$

$\dim(\mathbb{R}^3) = 3 \Rightarrow \dim(\ker A) = 1$

How to find a basis for the nullspace of a matrix

We are solving $A\vec{x} = \vec{b}$, where $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\text{rref}([A\ b]) \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$1 \cdot x_1 + 0 \cdot x_2 + 2 \cdot x_3 = 0$$
$$0 \cdot x_1 + 1 \cdot x_2 - 3 \cdot x_3 = 0$$
$$0 + 0 + 0 = 0$$

free variable: x_3

$$x_1 = -2x_3$$
$$x_2 = 3x_3$$
$$x_3 = 1 \cdot x_3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} x_3$$

$\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$ is a solution to $A\vec{x} = \vec{0}$

similarly $5\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$ solves $A\vec{x} = \vec{0}$

nullspace A = span $\left\{ \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} \right\}$

$\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$ is a basis for $\ker(A)$

Ex 2

$$A = \begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 3 & 2 & 5 & 2 \end{bmatrix}$$

$$M = [A \ \vec{0}] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

free variables: x_3, x_4

$$x_1 = -x_3$$
$$x_2 = -x_3 - x_4$$
$$x_3 = 1 \cdot x_3$$
$$x_4 = 1 \cdot x_4$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} x_4$$

basis for $\ker(A)$ is $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$