Lecture 9

<u>Review</u>

In $\mathbb{R}^{2\times 1}$, two linearly independent vectors generated the entire vector space

Case 1

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

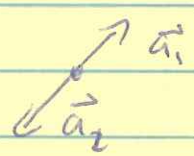
two non-parallel vectors can generate the entire $\mathbb{R}^2$ space. Therefore a solution to $A\vec{x} = b$ where $A = [\vec{a}_1, \vec{a}_2]$ always exists for any $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\text{span}(\vec{a}_1, \vec{a}_2) = W$, where $W = \mathbb{R}^2$

We can always find x_1, x_2 <u>uniquely</u> that satisfies the system for b_1, b_2

$$\vec{a}_1 x_1 + \vec{a}_2 x_2 = \vec{b}$$

Case 2



$\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ $\quad \text{span}(\vec{a}_1, \vec{a}_2) = U$ (subspace)

Let U consist of all vectors in $\mathbb{R}^2$ whose entries are s.t.

$$U = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : b = 2a \right\}$$ "such that"

therefore, we can only find a solution x_1, x_2 if $\vec{b} \in U \Rightarrow$ all $\vec{b}$'s s.t.

$b_1 = 2b_2$

$A\vec{x} = \vec{b}$

Linear spans

Suppose $\vec{u}_1, \dots, \vec{u}_n$ are any vectors in a vector space V. The collection of all linear combinations

$$a_1\vec{u}_1 + a_2\vec{u}_2 + \dots + a_n\vec{u}_n \quad \text{where } a_i \text{ are scalars}$$

denoted by

$$\text{span}(\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n) \text{ or } \text{span}(\vec{u}_i)$$

is called the linear span of $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n$

Zero vector belongs to the $\text{span}(\vec{u}_i)$, since

$$0 = 0\cdot\vec{u}_1 + 0\cdot\vec{u}_2 + \cdots 0\cdot\vec{u}_n$$

Theorem If $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n \in V$, then

(i) the $\text{span}(\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n)$ is a subspace of V that contains $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n$

(ii) If W is a subspace of V containing $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$ then

$$\text{span}(\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n) \subseteq W$$

"is a subset"

Rowspace of a Matrix

Let $A = [a_{ij}]$ be an arbitrary $m \times n$ matrix over a field $\mathbb{R}$. The rows of A,

$$\vec{r}_1 = (a_{11}\ a_{12}\ \dots\ a_{1n})$$
$$\vec{r}_2 = (a_{21}\ a_{22}\ \dots\ a_{2n})$$
$$\vdots$$
$$\vec{r}_m = (a_{m1}\ a_{m2}\ \dots\ a_{mn})$$

may be viewed as vectors in $\mathbb{R}^n$. They span a subspace of $\mathbb{R}^n$ called the row space of A and denoted by rowsp(A).

$$\text{rowsp}(A) = \text{span}(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_m)$$

[This is analagous to when we viewed the columns of A as vectors in $\mathbb{R}^m$, called the column space of A

$$\text{colsp}(A) = \text{span}(\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n)$$]

Note that $\text{colsp}(A) = \text{rowsp}(A^T)$

Example

If $A = \begin{bmatrix} 1 & 0 & 3 \\ 4 & -1 & 0 \end{bmatrix}$ then the transpose of A, denoted A^T is such that the i-th column becomes the i-th row

$$\Rightarrow A^T = \begin{bmatrix} 1 & 4 \\ 0 & -1 \\ 3 & 0 \end{bmatrix}$$

$$\text{colsp}(A) = \text{span}\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}\right)$$

$$\text{rowsp}(A^T) = \text{span}\left((1,4), (0,-1), (3,0)\right)$$

Theorem Row equivalent matrices have the same row space

Example

pivot points circled

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{bmatrix} \xrightarrow{\text{row canonical form}} \begin{bmatrix} 1 & 2 & 0 & 1/3 \\ 0 & 0 & 1 & -8/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Span}\{(1,2,-1,3),(2,4,1,-2),(3,6,3,-7)\} = \text{Span}\{(1,2,0,1/3),(0,0,1,-8/3)\}$$

Suppose we have another system w/ the same row canonical form

$$B = \begin{bmatrix} 1 & 2 & -4 & 11 \\ 2 & 4 & -5 & 14 \end{bmatrix} \xrightarrow[\text{row canonical form}]{} \begin{bmatrix} 1 & 2 & 0 & 1/3 \\ 0 & 0 & 1 & -8/3 \end{bmatrix}$$

$$\text{rowsp}(A) = \text{rowsp}(B)$$

Theorem The non-zero rows of a matrix in echelon form are linearly independent

Basis and Dimension

Two equivalent definitions of a basis

Def A: A set $S = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ of vectors is a basis of V if it has the following two properties:

(1) S is linearly independent

(2) S spans V

Def B: A set $S = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ of vectors is a basis of V if every $\vec{v} \in V$ can be written <u>uniquely</u> as a linear combination of the basis vectors

Example

Basis for $\mathbb{R}^2$ $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

or $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

A A vector space V is said to be n-dimensional, written $\dim V = n$ if V has a basis with n elements

A The vector space $\{0\}$ is defined to have dimension 0

<u>Lemma</u>: Suppose $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ Spans V, then a set of $n+1$ or more vectors in V are linearly dependent

If $\dim V = n$, then the entire vector space can be generated by n linearly independent vectors. If any additional vector $\vec{w}$ is added that is in V, then by definition $\vec{w}$ can be written as a linear combination of the set

$$\vec{w} = k_1\vec{v}_1 + k_2\vec{v}_2 + \ldots + k_3\vec{v}_n$$

So that

$\{\vec{w}, \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly dependent set

A Any basis of $\mathbb{R}^n$ has n elements

$\mathbb{R}^2$ has a basis of two vectors

$\mathbb{R}^3$ has a basis of three vectors

Example $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \leftarrow$ Standard basis

$\mathbb{R}^4$ has a basis of four vectors

$$\dim(\mathbb{R}^n) = n$$

Definition: The rank of a matrix A, written rank(A) is equal to the maximum number of linearly independent rows of A, or equivalently, the dimension of the row space of A

<u>Theorem</u> The maximum number of linearly independent rows of any matrix A is equal to ~~the~~ the maximum number of linearly idependent columns of A. Thus the dimension of the rowspace of A is equal to the dimension of the column space of A.