Lecture 9

Matrix-matrix product

$C = AB$ is defined as follows, if $a_{ij} = [A]_{ij}$ and $b_{ij} = [B]_{ij}$ the

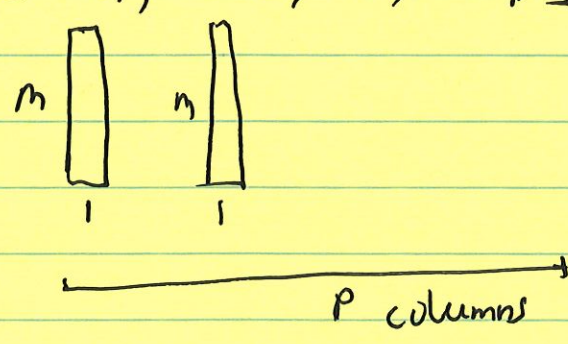
$$[C]_{ij} = [AB]_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$\underset{(m \times n)}{A} \; \underset{(n \times p)}{B} = \underset{(m \times p)}{C}$$

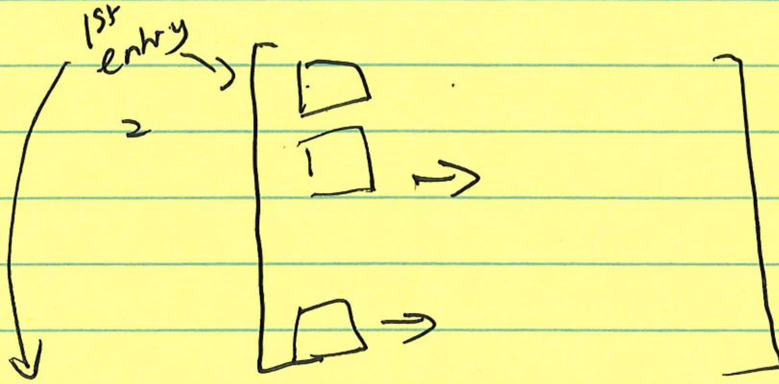
have to be the same

Suppose $B = [\vec{b}_1, \vec{b}_2, \vec{b}_3, \dots, \vec{b}_p]$ $\quad \vec{b}_i \in \mathbb{R}^{n \times 1}$

$$AB = [A\vec{b}_1, A\vec{b}_2, \dots, A\vec{b}_p]$$



$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1n} \\ a_{21} & \cdot & & \cdot \\ a_{22} & & \ddots & \\ \vdots & & & \\ a_{mn} & & & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & \cdot & & \\ \vdots & & \ddots & \\ b_{np} & & & b_{np} \end{bmatrix}$$



$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix} \begin{bmatrix} b_{11} \\ \vdots \\ b_{n1} \end{bmatrix} = a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1}$$

$$\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 9 & -2 \\ -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} \quad \vdots \quad \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Square Matrices $F_A : \mathbb{R}^n \to \mathbb{R}^n$

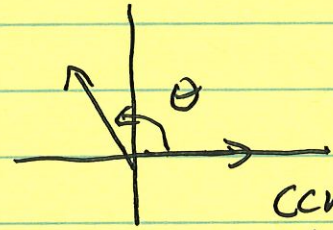
- Linear transformations
- Determinants
- Inverse

Types of transformations

Reflection, rotation, dilation, translation

Rotation

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

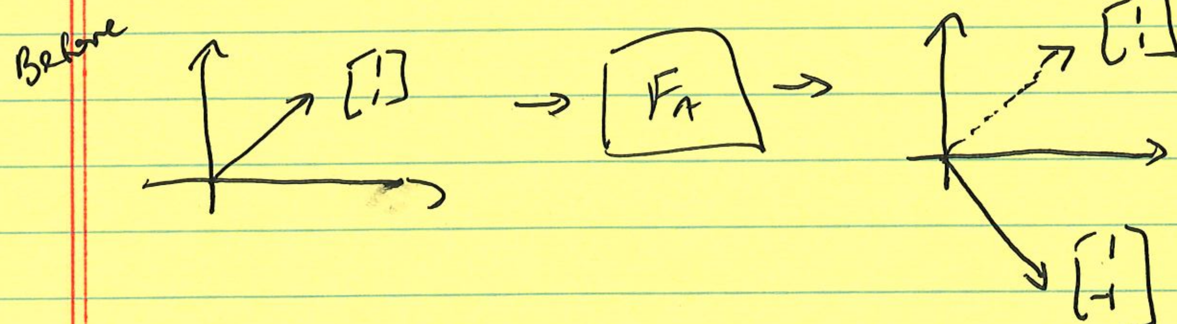


CCW rotation

Reflection

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

This will reflect a vector across the x-axis



AR : rotation + reflection

$y = Rx \to$ reflects it

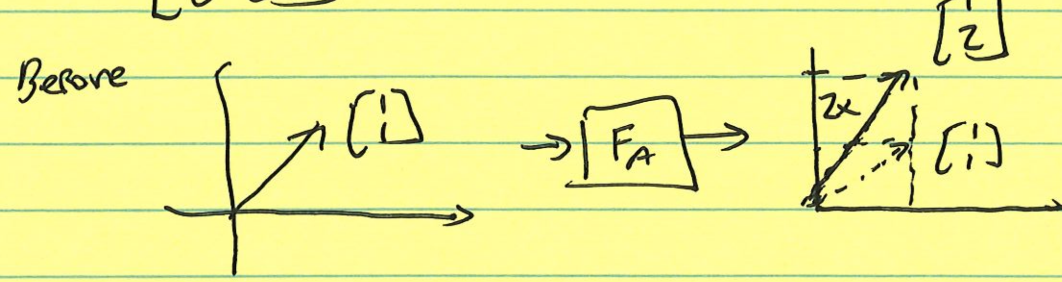
$u = Ay \to$ rotate

$u = \boxed{AR}x$

Dilation

Suppose I want to stretch my vector $2x$ in the y-direction

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

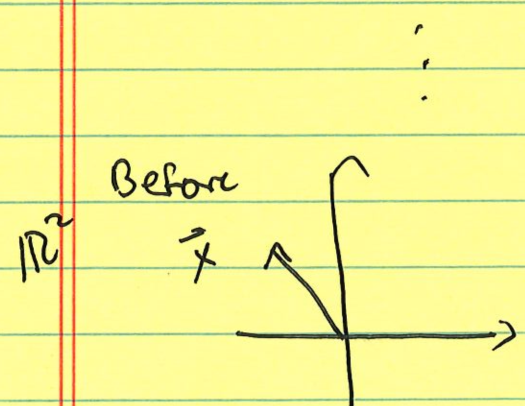


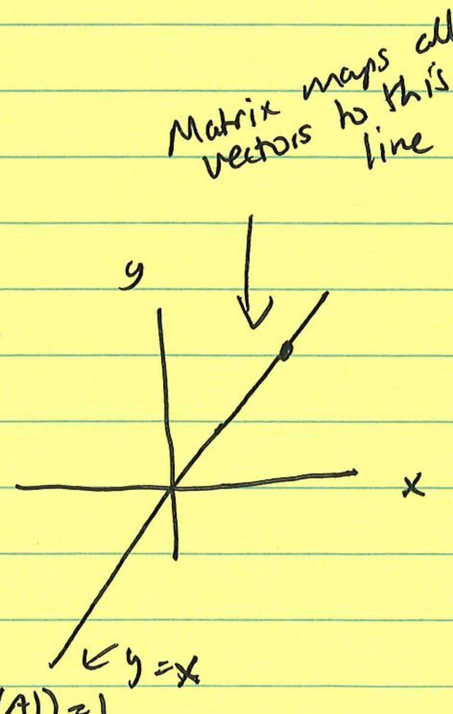
Let's look at a system where $A \in \mathbb{R}^{n \times n}$ and $\text{rank}(A) < n$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\vdots$





$\text{rank}(A) = 1$

$\dim(\text{Im}(A)) = 1$

$\dim(\mathbb{R}^2) = 2 \Rightarrow \dim(\text{Ker}(A)) = 1$

$A \in \mathbb{R}^{n \times n}$

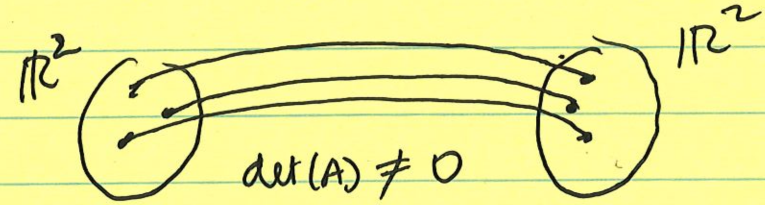
Two cases

$\text{rank}(A) = n$

$\text{rank}(A) < n$

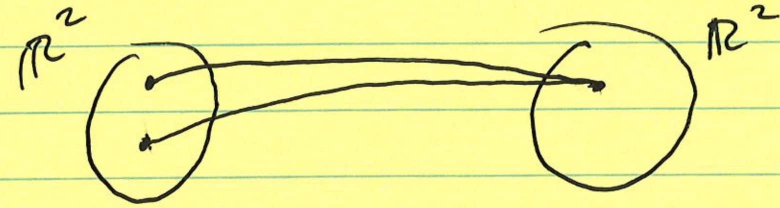
when the $\text{rank}(A) = n$

one - to - one map (onto)



There exists an invertible mapping A^{-1}

$\text{rank}(A) < n \Rightarrow \det(A) = 0$

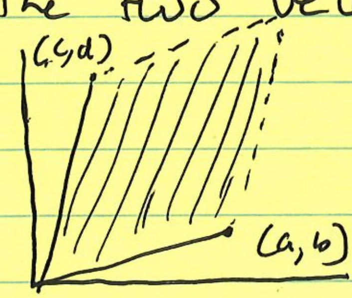


A^{-1} does not exist

Determinant

Determinant of 2×2 matrix gives the area spanned by the two vectors

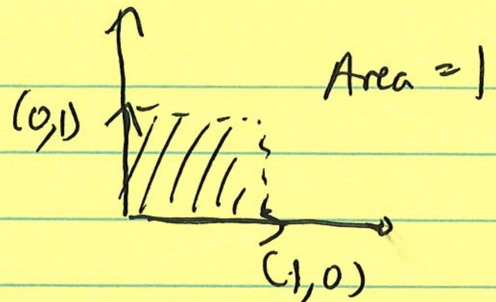
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$



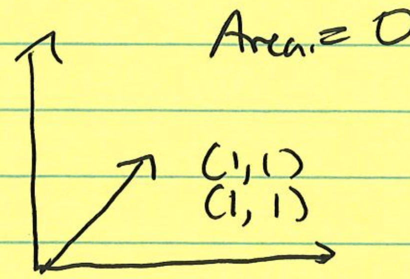
$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

Ex 1

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

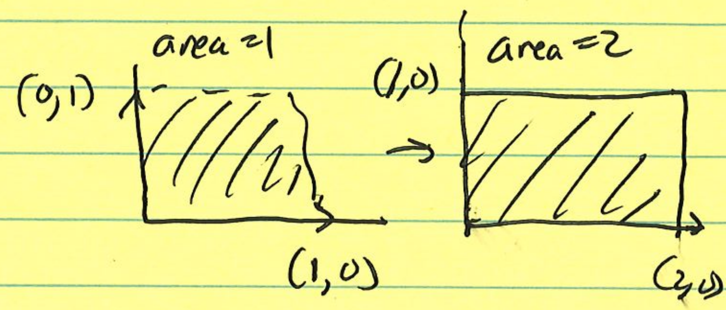


$$\det(A) = 1\cdot 1 - 1\cdot 1 = 0$$



Ex 2

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$



$$\det(A) = 2\cdot 1 - 0\cdot 0 = 2$$

doubling our area

Determinant of 3x3 matrix

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a(-1)^{1+1}\begin{vmatrix} e & f \\ h & i \end{vmatrix} + b(-1)^{1+2}\begin{vmatrix} d & f \\ g & i \end{vmatrix}$$

$i=1, j=1$

$$+ c(-1)^{1+3}\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Determinant of 4x4 matrix

See ppt... break up matrix iteratively into smaller sub problems

Properties of determinants

$$\det(AB) = \det(A)\det(B)$$

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\det(A) = 1 \cdot \begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} + 2 \cdot \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} + 2 \begin{vmatrix} 0 & 3 \\ 1 & 0 \end{vmatrix}$$

$$= 1 \cdot (3-0) + 2(0-2) + 2(0-3)$$

$$= 3 - 4 - 6 = -7$$

$$B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & -2 \end{bmatrix}$$

$$\det(B) = 2 \cdot \begin{vmatrix} -1 & -2 \\ 1 & -2 \end{vmatrix} + 0 + 1 \cdot \begin{vmatrix} 0 & -1 \\ 3 & 1 \end{vmatrix}$$

$$= 2(2+2) + 1(0+3)$$

$$= 8 + 3 = 11$$

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$$AB = \begin{bmatrix} 8 & 4 & 1 \\ 6 & -1 & -10 \\ 5 & 1 & -1 \end{bmatrix}$$

$$\det(AB) = -77$$

If A is an $n \times n$ matrix and c is a scalar, then

$$\det(cA) = c^n \det(A)$$

$$A = \begin{bmatrix} 10 & -20 & 40 \\ 30 & 0 & 50 \\ -20 & -30 & 10 \end{bmatrix} = 10 \underbrace{\begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{bmatrix}}_{B}$$

$n = 3$ $\det(B) = 5$

$$\det(A) = \det(cB) = 10^3 \det(B)$$
$$= 5{,}000$$

Inverse $\begin{pmatrix} A \in \mathbb{R}^{n \times n} \\ \text{rank}(A) = n \end{pmatrix}$

$$A^{-1} A = I$$

$I \in \mathbb{R}^{3\times3}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1}(A\vec{x} = \vec{b})$$

$$\underbrace{A^{-1}A}_{I}\vec{x} = A^{-1}\vec{b}$$

$$\Rightarrow \boxed{\vec{x} = A^{-1}\vec{b}}$$

2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} x_1 + \begin{bmatrix} 4 \\ 1 \end{bmatrix} x_2$$

Change basis to find equivalent representation in terms of standard basis

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} x_1 + \begin{bmatrix} 4 \\ 1 \end{bmatrix} x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} a_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} a_2$$

$$A\vec{x} = \vec{b} \Rightarrow [A : \vec{b}] \Rightarrow$$ echelon form

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Augmented matrix

$$\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right]$$

$$-2L_1 + L_2 \to L_2$$

$$\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & -7 & -2 & 1 \end{array}\right]$$

$$-\frac{1}{7}L_2 \to L_2$$

$$\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 1 & 2/7 & -1/7 \end{array}\right]$$

$$-4L_2 + L_1 \to L_1$$

$$\left[\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\;\middle|\;\underbrace{\begin{matrix} -8/7+1 & 4/7 \\ 2/7 & -1/7 \end{matrix}}_{A^{-1}}\right]$$

$$A^{-1}A = \begin{bmatrix} -1/7 & 4/7 \\ 2/7 & -1/7 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} 1 & -4 \\ -2 & 1 \end{bmatrix}$$

$$\det(A) = 1 \cdot 1 - 2 \cdot 4 = -7$$

$$A^{-1} = \frac{1}{-7} \begin{bmatrix} 1 & -4 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -1/7 & 4/7 \\ 2/7 & -1/7 \end{bmatrix}$$

Let's solve the system

$$A\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\vec{x} = A^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1/7 & 4/7 \\ 2/7 & -1/7 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/7 \\ 3/7 \end{bmatrix} \checkmark$$