Midterm Review Guide

Midterm Review Topics

- Be able to put complex numbers in exponential form
- Know how to calculate roots and powers of complex numbers
- Know how to compute matrix-vector and matrix-matrix products
- Reduction of matrices to echelon form
- Identify matrices in echelon form
- Identify pivot and free variables from a matrix in echelon form
- Know how to determine existence and types of solutions
- Reduction to row canonical form
- Calculation of matrix inverse
- Know how to use matrix inverse to solve a system of equations
- Understand linear dependence and independence (also in relation to linear combinations)
- Know how to find a basis for a span of a given set of vectors
- Find a basis for rowsp(A), colsp(A), and nullspace(A)
- Know how to determine the rank of a matrix
- Know how to determine the dim of a subspace spanned by a set of vectors
- Understand Rank-nullity theorem and how to deduce the dimension of the kernel of a matrix
- Know $\dim(\mathbb{R}^n) = n$
- Understand concepts of spans and subspaces
- Know how to determine if $span(u_1, \dots, u_m) = \mathbb{R}^n$
- Know conditions for existence of a matrix inverse

Additional Review Problems

Example 1 How to check if $\vec{b}$ is in the span of $a_1, a_2, \dots, a_n$. Hint: If it's in the span then there is a linear combination such that

$$b = k_1 a_1 + \cdots + k_n a_n.$$

Example 2 Is it possible that matrix $A \in \mathbb{R}^{11\times 9}$ has a pivot position in every row?
Example 3 Suppose $a_1, \dots, a_n \in \mathbb{R}^m$. How to check if span$(a_1, \dots, a_n)$=$\mathbb{R}^n$.
Example 4 How to check if $A\vec{x} = 0$ has a non-trivial solution (i.e. not $\vec{x} = 0$)
Example 5. Suppose $A \in \mathbb{R}^{11\times 15}$, does $A\vec{x} = 0$ have a non-trivial solution?
Example 6 How to check if a set of vectors is linearly independent.

Example 7. Under what conditions is $\{v_1, v_2, 0\}$ linearly dependent?
Example 8 Is $\{2\vec{u}, 7\vec{u}\}$ linearly dependent?
Example 9 Suppose $a_1, \ldots, a_n \in \mathbb{R}^m$ and $n > m$. Is the set linearly dependent?
Example 10 Suppose $A \in \mathbb{R}^{3\times 5}$ and $B \in \mathbb{R}^{4\times 3}$. Is AB well defined? Is BA well defined? Is A^7 well defined? Is $(AB)^7$ well defined?

Solutions

Example 1 How to check if $\vec{b}$ is in the span of $a_1, a_2, \ldots, a_n$. Hint: If it's in the span there there is a linear combination such that

$$b = k_1 a_1 + \cdots + k_n a_n$$

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Solution Construct a matrix A composed of the vectors as column vectors $A = [a_1, a_2, \ldots, a_n]$. Reduce the augmented matrix $M = [A, \vec{b}]$ to echelon form. If the reduced matrix does not result in an "inconsistent" row then it is in the span (it is inconsistent when there is a pivot point in the last column).

Example 2 Is it possible that matrix $A \in \mathbb{R}^{11\times 9}$ has a pivot position in every row?
Solution A matrix $A \in \mathbb{R}^{11\times 9}$ can have at most 9 linearly independent rows and so can have at most 9 pivot points. Therefore, there cannot be a pivot position in every row, if there are 11 rows.

Example 3 Suppose $a_1, \ldots, a_n \in \mathbb{R}^m$. How to check if span$(a_1, \ldots, a_n)$=$\mathbb{R}^n$.
Solution First, it is necessary that $m = n$, then you must check to see if the set of vectors are linearly independent. This means the matrix is full rank. This can be determined by reducing $A = [a_1, \ldots, a_n]$ to echelon form and counting the pivot points. The row canonical form will be a diagonal matrix.

Example 4 How to check if $A\vec{x} = 0$ has a non-trivial solution (i.e. not $\vec{x} = 0$)
Solution The system $A\vec{x} = 0$ has a nontrivial solution if the dim(Ker(A))>0. This means that more than just the zero vector in the domain maps to the zero vector in the image of A. This is true if the matrix is not full rank. In other words, the row/column vectors are linearly dependent.

Example 5. Suppose $A \in \mathbb{R}^{11\times 15}$, does $A\vec{x} = 0$ have a non-trivial solution?
Solution Assume the system is consistent. Then there exists a nontrivial solution. The column vectors are guaranteed to be linearly dependent. Since the number of rows are 11, that is the maximum possible number of linearly independent vectors. By definition, this means there exists a linear combination of the column vectors such that

$$k_1 a_1 + k_2 a_2 + \ldots k_n a_n = 0,$$

where not all coefficients $k_1, \ldots, k_n$ are zero.

Example 6 How to check if a set of vectors is linearly independent.
Solution Construct a matrix from the vectors and find the rank of the matrix. If the rank is less than the number of vectors then they are linearly dependent and independent otherwise.

Example 7. Under what conditions is $\{v_1, v_2, 0\}$ linearly dependent?
Solution Under all conditions. Any set including 0 is linearly dependent.

Example 8 Is $\{2\vec{u}, 7\vec{u}\}$ linearly dependent?
Solution Yes. One vector is a scalar multiple of the other.

Example 9 Suppose $a_1, \ldots, a_n \in \mathbb{R}^m$ and $n > m$. Is the set linearly dependent?
Solution Yes, because the number of vectors are more than dim($\mathbb{R}^m$)=m. A basis for $\mathbb{R}^m$ has m elements. A set with more than m elements must be linearly dependent.

Example 10 Suppose $A \in \mathbb{R}^{3\times 5}$ and $B \in \mathbb{R}^{4\times 3}$. Is AB well defined? Is BA well defined? Is A^7 well defined? Is $(AB)^7$ well defined?
Solution AB is not defined. $BA \in \mathbb{R}^{4\times 5}$. A^2 is not defined, so A^7 is not. Similarly, AB is not defined so $(AB)^7$ is not either.