1. True

$A \rightarrow$ reduce to echelon form $\rightarrow$ count pivot points

$\Downarrow$

rank(A)

2. $A \in \mathbb{R}^{3\times 2}$ False

maximum # of linearly independent row/column vectors is 2.

3. True

$Ax = 0 \leftarrow$ homogeneous

$x = 0$

4. True

$-v = (-1)v$

5. True

$\{u, u-v\}$

$v = -(u-v) + u$

$= v$ ✓

6. True

A has 4 linearly independent columns

rank$(A) = 4$

A has 4 linearly independent rows

7. False

colsp(A) = span of the column vectors

$\dim(\text{colsp}(A)) = \text{rank}(A) \leq 3$

8. True

$\det(A) = 0 \Rightarrow A^{-1}$ does not exist (DNE)

A^{-1} DNE $\Rightarrow$ A is <u>not</u> full rank

$\Rightarrow$ column vectors are <u>not</u> linearly independent

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \rightarrow$$

$\text{rank}(A) = \text{rank}(A^T)$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 3 \\ 4 & 4 & 1 \end{bmatrix} \Rightarrow \quad A^T = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 2 & 4 \\ 0 & 3 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

~~$\det(A^T) = \det(A$~~

9. False

If A has $\lambda = 0$

$\Rightarrow$ there exists x s.t. $Ax = 0$

$$\underset{\lambda=0}{Ax = \lambda x} \leftarrow \text{associated eigenvectors}$$

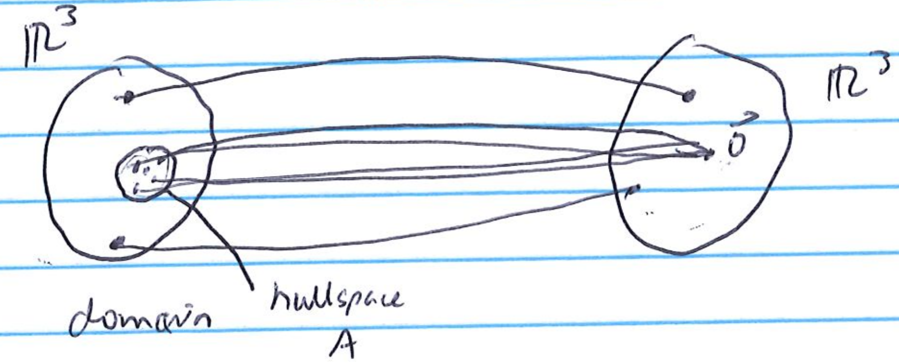
$\Rightarrow \dim(\text{Ker}(A)) > 0$

$\Rightarrow$ A is not full rank

$\Rightarrow$ not invertible

$A: \mathbb{R}^3 \to \mathbb{R}^3$

$Ax = b \qquad x \in \mathbb{R}^3 \qquad b \in \mathbb{R}^3$



11. False

Example consider

$$A = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ 0 & 0 & 0 & & \vdots \\ \vdots & & & 5 & 0 \\ 0 & \cdots & & 0 & 8 \end{bmatrix}$$

diagonalizable but not invertible

A is not full rank

10. False $\quad A \in \mathbb{R}^{n \times n}$

~~dia~~ diagonalizable matrix requires n linearly independent eigenvectors

Criteria for diagonalization
geometric multiplicity
= algebraic multiplicity
for all eigenvalues

12. False

$A \in \mathbb{R}^{m \times n} \qquad Ax = b$

$x \in \mathbb{R}^n$

13. False

$A \in \mathbb{R}^{m \times n}$ $\quad$ $\text{colsp}(A) \in \mathbb{R}^m$

14. False

Conditions on basis : linearly independent set

If A is not full rank then the column vectors are linearly dependent

15. True

If A is invertible, then the column vectors are linearly independent and there are n vectors in $\mathbb{R}^n$

$\Rightarrow$ this gives us a basis

16. True

Ker(A) is a set of all x s.t.

$Ax = 0$ $\rightarrow$ reduce to echelon form B

(row ~~eq~~ equivalent matrix)

$A \sim B$

$Bx = 0$ gives us solution to $Ax = 0$

17. False

two similar matrices have the same eigen values

$A = \underbrace{P}\,D\,\underbrace{P^{-1}}$ $\quad$ A is similar to D

Projection : Best Approximation Theorem / Least - Squares

Ex Find the closest point to y in the subspace spanned by v_1 and v_2, where

$$y = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix}, \quad v_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}, \quad \text{and } v_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}$$

Note that $v_1 \cdot v_2 = 0$

The closest point to y in $W = \text{span}\{v_1, v_2\}$ is $\text{proj}_W y = \hat{y}$

$$\hat{y} = \frac{y \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{y \cdot v_2}{v_2 \cdot v_2} v_2$$

$$= \frac{(1+0-3+4)}{(1+4+1+4)} v_1 + \frac{(-4+6)}{(16+1+9)} v_2$$

$$= \frac{2}{10} \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} + \frac{2}{26} \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -4 \\ -2 & 1 \\ -1 & 0 \\ 2 & 3 \end{bmatrix} \qquad y = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix}$$

$$[A \,\vdots\, y] \sim \left[\begin{array}{cc:c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right] \qquad \text{inconsistent}$$

What is the least-square solution to $Ax = y$

It's $\hat{x}$ s.t. $A\hat{x} = \hat{y}$ ← projection of y onto W

$$\hat{x} = \begin{bmatrix} \frac{2}{10} \\ \frac{2}{26} \end{bmatrix}$$

If no orthogonal basis is given

$$\hat{x} = (A^TA)^{-1} A^T y = \begin{bmatrix} 2/10 \\ 2/26 \end{bmatrix}$$

A^TA is only invertible when columns of A are linearly independent

$\Rightarrow$ unique least-square solution

A^TA is full rank

What is $W^\perp$

Let $W = \text{colsp}(A)$

Solve $A^T x = 0$

$$\begin{bmatrix} 1 & -2 & -1 & 2 \\ -4 & 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$4L_1 + L_2 \to L_2$

$$\begin{bmatrix} 1 & -2 & -1 & 2 \\ 0 & -7 & -4 & 11 \end{bmatrix}$$

free variables : x_3, x_4

$$\vec{x} = x_3 \begin{bmatrix} -1/7 \\ -4/7 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 8/7 \\ 11/7 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$-7x_2 = 4x_3 - 11x_4$$

$$x_1 = 2x_2 + x_3 - 2x_4$$

$$= 2\left(-\frac{4}{7}x_3 + \frac{11}{7}x_4\right) + x_3 - 2x_4$$

$$= -\frac{1}{7}x_3 + \frac{8}{7}x_4$$

$$W^\perp = \text{span}\left\{ \begin{bmatrix} -1/7 \\ -4/7 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 8/7 \\ 11/7 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\dim(W^\perp) = 2$$

$$\text{Let} \quad y = \overset{\swarrow\ \text{proj}_W y}{\hat{y}} + \text{proj}_{W^\perp} y$$

What is $A^T(y - \hat{y}) = ?$

$$A^T(y - \hat{y}) = 0$$

$$y - \hat{y} = \text{proj}_{W^\perp} y$$

Practice Questions

Q50

$\tau = 3.089$ is an eigenvalue of A

$$\underbrace{(A - \tau I)}_{M} x = 0$$

if $Mx = 0$ ~~has~~ has non-zero solution

$\Rightarrow \dim(\cancel{n}\ \ker(M)) > 0$

$\Rightarrow$ M has eigenvalue equal zero

$$\Rightarrow \det(M) = \tau_1^M \tau_2^M \cdots \tau_n^M = 0$$

$$\det(A - \tau I) = 0$$
$$\det(\tau I - A) = 0$$

$$Mx = \tau x \quad \cancel{\Rightarrow \text{now}}$$
$$= 0 \quad \Rightarrow \tau = 0$$

Complex Numbers

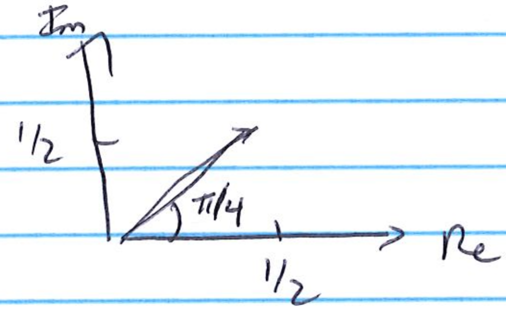
$$z = \frac{2+i}{1-i} + \frac{1-i}{1+i}$$

$$= \frac{2+i}{1-i} \cdot \frac{(1+i)}{(1+i)} + \frac{(1-i)}{(1+i)} \cdot \frac{(1-i)}{(1-i)}$$

$$= \frac{2+3i-1}{1+1} + \frac{1-2i-1}{1+1}$$

$$= \frac{1+i}{2} = \frac{1}{2} + \frac{1}{2}i$$

$Re(z) = \frac{1}{2}$ $\qquad$ $Im(z) = \frac{1}{2}$



$\theta = \pi/4$

$z = re^{i\theta}$ $\qquad$ $r = \sqrt{Re^2 + Im^2}$

$$= \sqrt{\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2}$$

$$= \sqrt{\tfrac{1}{4} + \tfrac{1}{4}} = \sqrt{\tfrac{2}{4}} = \frac{\sqrt{2}}{\sqrt{4}}$$

$$= \frac{\sqrt{2}}{2}$$

$$z = \frac{\sqrt{2}}{2} e^{i\pi/4}$$

Find $\mathrm{Re}(z^{-12})$

$$z^{-12} = \left(\frac{\sqrt{2}}{2}\right)^{-12} \left(e^{i\pi/4}\right)^{-12}$$

$$= \frac{2^{-6}}{2^{-12}} e^{i\pi/4(-12)}$$

$$= 2^{6} e^{-i3\pi}$$

$$= 2^{6} e^{-i\pi}$$

$$= 2^{6}\left(\cos(-\pi) + \cancel{i\sin(-\pi)}^{\,0}\right)$$

$$= 2^{6}(-1)$$

$$= -2^{6}$$

$\mathrm{Re}(z^{-12}) = -2^{6}$ $\quad \mathrm{Im}(z^{-12}) = 0$

Find $z^{1/2}$ $\Rightarrow$ 2 roots

$$z^{1/2} = \left(\frac{\sqrt{2}}{2}\right)^{1/2} \left(e^{i(\pi/4 + 2\pi k)}\right)^{1/2} \quad \begin{matrix} k=0 \\ k=1 \end{matrix}$$

$$z^{1/2} = \left(\frac{\sqrt{2}}{2}\right)^{1/2} e^{i(\pi/4 + 2\pi k)\frac{1}{2}}$$

$$= \left(\frac{\sqrt{2}}{2}\right)^{1/2} e^{i(\pi/8 + \pi k)} \quad \begin{matrix} k=0 \\ k=1 \end{matrix}$$

two solutions

$k=0$ $\quad z = \left(\frac{\sqrt{2}}{2}\right)^{1/2} e^{i\pi/8} = \frac{2^{1/4}}{2^{1/2}} e^{i\pi/8}$

$k=1$ $\quad z = \left(\frac{\sqrt{2}}{2}\right)^{1/2} e^{i(\pi/8 + \pi)}$

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$$z^{1/2} = \left(\frac{\sqrt{2}}{2}\right)^{1/2} e^{i(\pi/4 + 2\pi k)\frac{1}{2}}$$

$$= \left(\frac{\sqrt{2}}{2}\right)^{1/2} e^{i(\pi/8 + \pi k)} \quad \begin{matrix} k=0 \\ k=1 \end{matrix}$$

two solutions

$k=0 \quad z = \left(\frac{\sqrt{2}}{2}\right)^{1/2} e^{i\pi/8}$

$k=1 \quad z = \left(\frac{\sqrt{2}}{2}\right)^{1/2} e^{i(\pi/8 + \pi)}$